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DETERMINATION OF THE EQUILIBRIUM SHAPE OF THE BODIES FORMED DURING THE SOLIDIFICATION OF FILTRATION FLOW*

K.G. KORNEV and V.A. CHUGUNOV

It is shown that the problem of determining the equilibrium shape of the bodies formed when a filtration flow solidifies, can be reduced to the Riemann problem with shear. A solitary body is used as an example, and an algorithms for determining its boundary is constructed and realized. The qualitative properties of the solution of the problem in question are studied.

1. Formulation of the problem.

The method of freezing water-laden rocks is widely used in building various types of constructions /1/. The process of solidifying a filtration flow around a cold source is characterized by the fact that after a time a thermal balance is reached. The heat flux densities at the phase boundary become equal, and this means that the rate of formation of the solid becomes equal to zero. Thus the shape of the solid formed when the filtration flow solidifies reaches, in time, its limiting form, which we shall call the equilibrium form.

If we assume that the process takes place in the plane $z = x + iy$, that the filtration obeys D'Arcy's law, that the fluid is incompressible and that the thermophysical characteristics of the filtering medium are constant, the mathematical model of the phenomenon in question can be represented in the form

$$\mathbf{v} = -k\nabla p, \quad \operatorname{div}\mathbf{v} = 0, \quad K_c\mathbf{v}\nabla t = a_+\Delta t \quad z \in D \tag{1.1}$$
$$\Delta t_k = 0, \quad z \in D_k$$

$$\mathbf{v} \to v_\infty, \quad t \to t_\infty, \quad |z| \to \infty \tag{1.2}$$

$$\lambda_+ \partial t/\partial n = \lambda_- \partial t_k/\partial n, \quad t = t_k = t_*, \quad z \in \partial D_k \tag{1.3}$$

$$t_k = t_0 < t_*, \quad z \in \Gamma_k \tag{1.4}$$

Here D is the region of filtration, D_k is the region occupied by the solid, ∂D_k is its boundary, $\mathbf{v}$ is the rate of filtration, p is pressure, k is the coefficient of filtration, t, t_k are the temperatures in the region D and D_k respectively, K_c is the ratio of the heat capacities of the liquid and the filtering medium, λ_+, λ_- are the thermal conductivities in the regions D and D_k respectively, a_+ is the thermal diffusivity in D, n is the normal to the surface ∂D_k, external with respect to the region D_k, t_* is the temperature at

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which the flow solidifies, t_∞, v_∞ is the temperature and velocity at the point at infinity, and Γ_k is a given surface on which a given temperature $t_0 < t_*$ is maintained.

It should be noted that all subsequent arguments remain valid if instead of (1.4) we specify the intensity of the cold source.

2. Passage to the Riemann problem with shear. The first two equations of (1.1) enable us to introduce, in a standard manner, the complex potential of the flow $W = \varphi + i\psi$ where φ is the velocity potential and $\varphi = -kp, \psi$ is the stream function.

Let us apply the Boussinesq transformation to the third equation of (1.1), which is equivalent to a conformal mapping of the physical z plane onto the potential W plane. The solids formed in the z plane will have corresponding cuts in the W plane, parallel to the φ axis. This simplifies the equation in question, and the determination of the temperature reduces to solving the problem

$$\frac{K_c}{a_+}\frac{\partial t}{\partial \varphi} = \frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial \psi^2}; \quad t \to t_\infty, \quad |W| \to \infty$$
$$t = t_*, \quad W \in L_k, \quad L_k = \bigcup_k \{\psi = \psi_k, \varphi \in [a_k, b_k]\} \tag{2.1}$$

The parameters a_k, b_k are governed by the values of the velocity potential φ at the critical points of the streamlined bodies. It should be noted that when a_k, b_k are specified, system (2.1) becomes closed by the last equation. Using (1.1), we introduce the complex heat potential $W_k = -t_k + i\psi_k$ where ψ_k is a function of the heat flux. Then the first condition of (1.3) on the unknown boundary ∂D_k will be transformed to

$$\lambda_+ / \lambda_- \mid \partial t/\partial\psi \mid \mid dW/dz \mid = \mid dW_k/dz \mid \tag{2.2}$$

Moreover, by virtue of the last two relations of (1.3), we can conclude that the boundary ∂D_k sought is an isotherm. Therefore, the heat flux density vector will be directed along the normal to ∂D_k. On the other hand, the curve ∂D_k serves as a stream line, and hence the velocity vector is directed along the tangent to ∂D_k.

Taking into account what has been said, and denoting by A_k, B_k the leading and trailing stagnation points of the body, we can write

$$\arg \frac{\overline{dW}}{dz} = \arg \frac{\overline{dW}_k}{dz} + (-1)^n \frac{\pi}{2}, \quad n = \begin{cases} 0, & z \in \widehat{A_k B_k} \\ 1, & z \in \underset{\smile}{A_k B_k} \end{cases} \tag{2.3}$$

Conditions (2.2) and (2.3) enable us to formulate the initial problem in the form of a problem of conjugation with discontinuous coefficients on a system of closed contours. Indeed, introducing the function

$$\Phi = \begin{cases} \Phi^+ = dW/dz, & z \in D \\ \Phi^- = dW_k/dz, & z \in D_k \end{cases}$$

and taking into account (2.2) and (2.3), we arrive at the following problem: to find the functions Φ^+, Φ^- analytic in the regions D and $\bigcup_k D_k$ respectively, satisfying the following linear relation on the contour $L = \bigcup_k \partial D_k$:

$$\Phi^- = G\Phi^+, \quad G = i(-1)^n (\lambda_+/\lambda_-) \mid \partial t/\partial\psi \mid \tag{2.4}$$

and conditions which follow from (1.4).

It is difficult to solve this problem in the physical plane, since the contours on which the boundary conditions are specified are not known in advance. We shall therefore employ the method of parametrization, which is widely used in the theory of ideal fluid flows /2, 3/.

Let us consider the plane of canonical variable ω. The lines separating the phases have corresponding canonical curves in this plane, whose form is chosen from the point of their subsequent suitability. Let us map the region D into the exterior of the corresponding contours of the ω plane. The feasibility of such mapping follows from /4/. We shall denote the mapping function by $z^+(\omega^+)$. We shall map the region D_k into the interior of the corresponding contour $\partial\Omega_k$ by means of the function $z^-(\omega^-)$. If $z \in \partial D_k$, then on approaching this point from region D we have $\omega^+ \to \xi \in \partial\Omega_k$ and if we approach z from D_k we have $\omega^- \to \eta \in \partial\Omega_k$. In the general case $\xi \neq \eta$. Condition (2.4), however, enables us to establish the relation $\eta = \eta(\xi)$ (which is called shear in /5, 6/). Indeed, we can rewrite condition (2.4) in the form

$$\frac{dW_k}{d\omega^-}(\eta)\chi^-(\eta) = \frac{dW}{d\omega^+}(\xi)\chi^+(\xi)G(\xi), \quad \chi^\pm = \frac{d\omega^\pm}{dz} \tag{2.5}$$

Then

$$\int_{\eta_0}^{\eta} \frac{dW_k}{d\omega^-}(\eta)\, d\eta = \int_{\xi_0}^{\xi} \frac{dW}{d\omega^+}(\xi)\, G(\xi)\, d\xi \tag{2.6}$$

It is obvious that expression (2.6) defines the shear $\eta = \eta(\xi)$. It should be noted that the function $dW_k/d\omega^-$, $dW/d\omega^+$ can be found using the methods of the theory of jets /2, 3/.

Thus we have reduced the initial non-linear problem (1.1)-(1.4) to the Riemann problem with shear (2.4), (2.6). The method of solving it is discussed in detail in /5, 6/. It should be stressed that such transformation does not eliminate the non-linearity of the problem, it merely transfers it to the process of determinating the parameters connected with the geometrical and physical characteristics of the problem, which appear when conformal mappings are introduced.

The solution of problem (2.5) yields the required boundary ∂D_k by simple integration

$$z = \int_{\eta_0}^{\eta} \frac{d\eta}{\chi^-(\eta)} + C \tag{2.7}$$

3. Solidifying of filtration flow around an axial cold source. We shall illustrate the above scheme of determining the free boundary using the following specific example.

We place the origin of coordinates of the z plane at the point where the cold source of intensity q is situated, and direct the x axis along the flow. It is clear that in this case an isolated solid will form, and therefore $k = 1$. We reduce system (1.1)-(1.4) to dimensionless form by putting /7/

$$X = \frac{x}{l}, \quad Y = \frac{y}{l}, \quad \theta = \frac{t - t_*}{t_\infty - t_*}, \quad \theta_1 = \frac{2\pi\lambda_-(t_1 - t_*)}{q}$$
$$\mathrm{Pe} = K_c \frac{v_\infty l}{a_+}, \quad \mathbf{V} = \frac{\mathbf{v}}{v_\infty}$$

(l is the characteristic size of the solid formed and Pe is the Peclet number). Using the methods of the theory of dimensions, we establish that

$$l = \left[\frac{q}{2\pi\lambda_+(t_\infty^- - t_*)}\right]^2 \frac{a_+}{K_c v_\infty} \tag{3.1}$$

We can use the formula (3.1) for preliminary estimations of the size of the body formed.

We take the ω plane as the domain of the canonical variable with unit circle whose centre lies at the origin of coordinates (Fig.1 shows the correspondence of these points). Then the complex flow potential will be given in canonical variables by the Zhukovskii potential

$$W = a(\omega^+ + 1/\omega^+) \tag{3.2}$$

where a is a parameter to be determined. The complex heat potential in canonical variables can be determined using the method of singular points /3/

$$W_1 = -\ln \omega^- \tag{3.3}$$

Problem (2.1) for determining the functions θ and G (the latter is needed to formulate the Riemann problem) takes the form

$$\mathrm{Pe}\, \partial\theta/\partial\varphi = \Delta\theta; \quad \theta \to 1, \quad |W| \to \infty \tag{3.4}$$
$$\theta = 0, \quad W \in L_1 = \{\psi = 0, \ \varphi \in [-2a, 2a]\}$$

A solution of this problem exists and can be obtained in the form of series in terms of Mathieu and Airy functions /8, 9/.

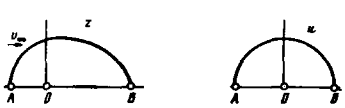

Fig.1

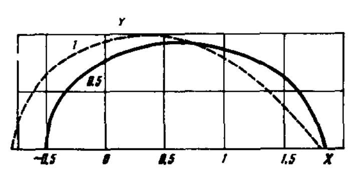

Fig.2

It is worth noting here that problem (3.4) is equivalent to the integral equation /7, 8/

$$\pi = \int_{-1}^{1} \mu(\xi) K_0(a\,\mathrm{Pe}\,|t-\xi|)\exp[-a\,\mathrm{Pe}(\xi - t)]\,d\xi \tag{3.5}$$

$$\mu = 2a\frac{\partial\theta}{\partial\psi}\Big|_{\psi=0,\ \varphi=2a\xi}, \quad t = \frac{\varphi}{2a}$$

where ($K_0(z)$ is the MacDonald function. Analysis of this equation yields the following asymptotic formulas (γ is Euler's constant) /7, 10/

$$\mu(\xi) = \frac{A}{2\pi\sqrt{1-\xi^2}} + O(4\beta^2\ln(2\beta)) \tag{3.6}$$

$$A = \pi[\gamma - \ln(4\beta)]^{-1} + O(4\beta^2\ln(2\beta)); \quad \beta = a\,\mathrm{Pe}\to 0$$

$$\mu(\xi) = \sqrt{\frac{2\beta}{\pi(1+\xi)}} + \sqrt{\frac{2\beta}{\pi(1-\xi)}}\,\frac{e^{2\beta\xi}K_0(2\beta)}{2\pi} - \tag{3.7}$$

$$\frac{2}{\pi}\sqrt{2\beta}\,e^{-2\beta(1-\xi)}\int_0^\infty e^{-2\tau^2}\operatorname{erfc}\sqrt{(1-\xi)(\tau^2+2\beta)}\,d\tau + O(e^{-2\beta});$$

$$\beta\to\infty$$

Calculations show that formula (3.7) can be used over a wide range of variation of the parameter β.

The equation for determining the quantity a follows from the equation of heat balance. Indeed, the amount of heat absorbed by the source must be equal to the amount of heat taken from the boundary ∂D_1. Therefore, if we use the boundary conditions on ∂D_1, then

$$\frac{1}{2\pi\sqrt{\mathrm{Pe}}}\int_{\partial D_1}\frac{\partial\theta}{\partial n}\,ds = 1$$

Passing in the given interval to coordinates φ, ψ and recalling the definition of the function $\mu(\xi)$, we can transform the last relation to the form

$$\int_{-1}^{1}\mu(\xi)\,d\xi = \pi\sqrt{\mathrm{Pe}} \tag{3.8}$$

i.e. we obtain the equation for the unknown β.

Using the maximum principle for the problem of the type (3.4) /11/ and equation (3.5), we can show that the integral on the left-hand side of expression (3.8) is a monotonically increasing function of the parameter β. Moreover, from (3.6) and (3.7) it follows that the magnitude of this integral approaches zero as $\beta\to 0$, and infinity as $\beta\to\infty$. Thus Eqs.(3.8) has a unique solution for the parameter β for any values of $\pi\sqrt{\mathrm{Pe}}$. Knowing the parameter β, we can also find the quantity $a = \beta/\mathrm{Pe}$.

We use formula (2.6) to determine the shear

$$\eta = e^{i\alpha(\sigma)}, \quad \xi = e^{i\sigma}, \quad \alpha(\sigma) = \int_0^\sigma f(\tau)\sin\tau\,d\tau \tag{3.9}$$

$$f(\tau) = \pm\mu(\cos\tau)/\sqrt{\mathrm{Pe}}$$

where the plus sign is taken when $0\leqslant\tau<\pi$, and the minus sign when $\pi\leqslant\tau\leqslant 2\pi$.

It is clear that the integrand in the third expression of (3.9) is positive within the interval $[0, 2\pi]$ in question. Therefore, the relation $\alpha = \alpha(\sigma)$ is monotonic and there exists a function inverse to $\eta(\xi)$, which we shall denote by $\eta^{-1}(\xi)$.

Taking into account (3.2), (3.3) and (3.9), we shall write Eq.(2.5) in the form ($\partial\Omega_1$ is the circle of unit radius)

$$\chi^-(\eta(\xi)) = \chi^+(\xi)G^*(\xi), \quad \xi\in\partial\Omega_1 \tag{3.10}$$

$$G^*(\xi) = f(\sigma)\sin\sigma e^{i(\alpha-\sigma)}$$

It should be noted that $G^*(\xi) = \eta'(\xi)$.

Thus it is required to determine the functions χ^-, χ^+, analytic inside and outside the contour $\partial\Omega_1$ respectively, satisfying on $\partial\Omega_1$ the condition (3.10). This the well-known Gaseman problem /5, 6/. In the case when $G^* = \eta'(\xi)$, it has a unique solution given by

$$\ln \chi^- = \frac{1}{2\pi i} \int\limits_{\partial\Omega_1} \frac{F(\eta^{-1}(\xi))\, d\xi}{\xi - \omega} + \ln \frac{1}{a}, \quad \omega \overline{\in} \Omega_1 \tag{3.11}$$
$$\ln \chi^+ = \frac{1}{2\pi i} \int\limits_{\partial\Omega_1} \frac{F(\xi)\, d\xi}{\xi - \omega} + \ln \frac{1}{a}, \quad \omega \in \Omega_1$$

where $F(\xi)$ is the solution of the Fredholm integral equation

$$F(\xi) + \frac{1}{2\pi i} \int\limits_{\partial\Omega_1} \left[\frac{\eta'(\tau)}{\eta(\tau) - \eta(\xi)} - \frac{1}{\tau - \xi}\right] F(\tau)\, d\tau = \ln G^*(\xi) \tag{3.12}$$

Having found the functions χ^-, χ^+, we can write the equation for the unknown boundary ∂D_1, in quadratures (2.7):

$$z = \int\limits_0^\sigma \frac{ie^{i\tau}\, d\tau}{\chi^+(e^{i\tau})} + \int\limits_0^1 \frac{d\tau}{\chi^-(\tau)} \tag{3.13}$$

Thus we have established that the problem of determining the boundary of a solid formed around a single axial cold source, has a unique solution determined by relation (3.3). Determining the coordinates of the points of the contour in question reduces to solving Eq.(3.12) and evaluating the integrals appearing in formulas (3.11) and (3.13).

The integral Eq.(3.12) was solved using the method of collocations. Fig.2 shows the results of the computations carried using the algorithm described above, for $\mathrm{Pe} = 2, 3$ (the solid line). The dashed line is the curve constructed in /12/. We see that the essential difference between the contours constructed is the presence of a sharp edge in the boundary obtained in /12/. In fact, experimental data show /13/ that the boundary of the body formed should be smooth. The appearance of a sharp edge in the solution in /12/ can be explained by the fact that it does not take into account the requirement that the density of heat flux at the critical points should be separable from zero /7/.

4. Estimation of the area occupied by the body formed.

The following assertion is of help in the operative estimation of the area of the body composed of ice and rock: of all bodies bounded by the zero isotherm, with the total heat transfer from the filtration flow given, the circle has the largest area.

The proof of the above assertion is based on the properties of conformal mapping. Let us denote by $T(z)$ the function which maps the outside of the solidified body onto the outside of a circle of radius r, so that $T(\infty) = \infty$ and $dT/dz|_\infty = 1$. The quantity r is called the outer radius of the region in question in the z plane /14/. The outer radius is directly related to the length of the cut in the W plane. Indeed, since the relation connecting W with T is given by the expression $W = T + r^2/T$ and the length of the cut in the W plane is equal to $4a$, it obviously follows that $r = a$. The quantity a, and hence r, can be uniquely determined by specifying the rate of heat flow (3.8). We know from the theory of isoperimetric inequalities /14/ that $S \leqslant \pi r^2$, and the equality is attained on a circle, which completes the proof.

In conclusion we note that in Sect.3 we have considered only one special case. The general scheme of determining the equilibrium form of the bodies which form when the filtration flow solidifies, presented in this paper, also covers the cases of many bodies. Moreover, the scheme makes possible not only the construction of an algorithm for determining the unknown boundaries, but it also helps to determine a number of qualitative properties of the solution of the problem in question.

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MULTIMODE BIFURCATIONS OF ELASTIC EQUILIBRIA*

YU.I. SAPRONOV

Conservative elastic systems with parallelepiped symmetry are considered, for which a study of the postcritical equilibria reduces (by the Lyapunov-Schmidt method) to the analysis of extremals of functions of the form

$$W(x_1, \ldots, x_n, \lambda) = \Sigma\alpha_j(\lambda)\, x_j^2 + \Sigma h_{i,j} x_i^2 x_j^2 + \ldots$$

where $H = (h_{i,j})$ is a symmetric matrix with non-degenerate principle (diagonal) minors. A relationship is written down for which the matrix H is determined by Ritz approximations of the total energy functional constructed by means of the fundamental bifurcation modes. In the case of soft buckling and for $\operatorname{ind} H = 0$ or $n-1$ (ind is the number of negative eigenvalues taking multiplicity into account) all the allowable types and quantities of bifurcating stable equilibriums are listed. It is shown that after soft buckling with breaking of symmetry, cascade bifurcations are possible (cascade bifurcations simulate the postcritical series of snappings accompanied by a drop in the load /1, 2/). Two known examples of soft buckling and one new example of hard buckling are presented for illustration.

Multimode bifurcations of elastic equilibria were investigated on the basis of a variational (energetic) principle within the framework of problems of the postcritical behaviour of elastic systems /1-3/. The fundamental achievements are obtained here under the influence of the theory of singularities of smooth functions /4, 5/ and ideas associated with the symmetry condition (equivalence of equilibrium equations relative to the action of a group in configuration space) /6-10/. It should be noted that the majority of the results associated with equivalence with respect to a continuous group are obtained by reduction (factorization by means of the group action orbits) to a single-mode bifurcation.

The oft-encountered symmetry of a parallelepiped (equivalence with respect to the action of a group $(Z_2)^n = Z_2 \times \ldots \times Z_2)$) results in the analysis of a function that is even in each variable, or equivalently, in the analysis of a function in a cone $R_+^n = \{x \in R^n \mid x_j \geqslant 0\}$ /3, 10, 11/. Up to now, bimodal bifurcations (with the symmetry of a rectangle) reducible to an analysis of functions of the form /5, 12/ $\alpha_1 x_1^2 + \alpha_2 x_2^2 + x_1^4 + a x_1^2 x_2^2 + x_2^4$, $a^2 \neq 4$ have been **investigated** practically completely. In the case of n modes $(n \geqslant 3)$, it has been established for bifurcations reducible to the analysis of functions of the form $(\alpha, y) + (Hy, y) + \ldots$, $y = (x_1^2, \ldots, x_n^2)^T$ (under the condition of degeneracy of the principal minors of H) that the number of orbits

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